Approximation solution of 0D pulsatile flow within the capillary

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Abstract

This paper deals with weak solution of the momentum balance equations. Different shapes of velocity profile are taken into account for description of pulsatile flow. The flow is assumed incompressible within the rigid capillary tube with constant diameter. The results of approximated solution of flow are compared with exact solution for various Womersley numbers. The same analysis was performed for a different type of weight functions and weighted residual methods. The assumption of Hagen-Pouseuille velocity profile cause error of flow rate in tens of per cent in the range of Womersley numbers 1-12 while the error of flow rate computed from fourth order polynomial velocity profile is only a few percents. The analysis proved that the integration of the residual and weight over the cross-section of pipe provides results of flow rate closer to the exact solution then the integration over radius. The integration over the area set the importance on wall function and back flow rate near the wall. It was revealed that the Galerkin method is the appropriate method for formulation of the weak solution of the pulsatile flow then the least square method and expert estimation of weighted function.

Keywords

Fluid transient, pressure pulsation, pulsatile flow, weak formulation

1. Introduction

The previous work [1] deals with natural oscillation of water column within the rigid tube. The error between measured frequency and computed from mathematical model was with assumed Hagen-Pouseuille flow approximately 24% and for flow computed from fourth order polynomial velocity profile 16%. It seems that the shape of velocity profile has significant influence on natural frequency of oscillated water column therefore we decide to study how accurately the flow is computed when the approximate solution is adopted.

The 0D models also known as windkessel or lumped parameter models are often used for description flow at arterial tree. The overview of 1D and 0D model that can be found in [2] allowed us to identify several approximation of velocity profile (flat, Hagen- Pouseuille, power law) that are adopted for description of 0D pulsatile flow. The approximation of velocity profile with assumed boundary layer was introduced in [3] previously formulated by Bessems [4] in 1D case. The analytical solution of pulsatile flow for harmonic pressure gradient in rigid tube was developed first by Womersley [5]. Luchini et al. [6] presented formulation based on approximate solution of kinetic balance equation for non-stationary flow.

2. Methods

2.1 Mathematical model – Weak formulation

Pulsatile flow of incompressible Newtonian liquid in a rigid pipe with circular and constant cross section is described by Eq.(1)

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{1}$$

where μ is the dynamic viscosity. The pressure gradient can be considered as a known function $P(t)$ if the 0D formulation is assumed.

$$P(t)=\frac{\partial p}{\partial x} \tag{2}$$

The velocity profile $u(r,t)$ can be represented as series of polynomial basis functions

$$u(r,t)=\sum_{i=1}^{n}U_i(t)N_i(r)=\sum_{i=1}^{n}U_i(t)\left(\frac{r^{2(i-1)}}{R^{2(i-1)}}-\frac{r^{2i}}{R^{2i}}\right) \tag{3}$$

The basis functions were selected in order to be symmetric with respect to the longitudinal axis of flow and fulfilled the boundary conditions ($u(R,t)=0,\ \frac{\partial u(0,t)}{\partial r}=0$). The first term $i=1$ corresponds to the parabolic function (Hagen Poiseuille) the further terms with higher power exponent can be called wall functions that allow back flow near the wall.
The Galerkin weighted residual method is employed to the derivation of approximated solution (4). The integration of the residual res and weight function w_j is over area $2\pi r dr$ but it is possible to integrate over radius dr.

$$\begin{aligned}\int_0^R res\cdot w_j dr &= \int_0^R\left[\rho\frac{\partial u}{\partial t}+P-u\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]N_j(r)2\pi r dr=\\ &=\rho\sum_{i=1}^{n}\frac{\partial U_i(t)}{\partial t}\underbrace{\int_0^R N_i(r)N_j(r)2\pi r dr}_{\mathbf{M}}+P\underbrace{\int_0^R N_j(r)2\pi r dr}_{\mathbf{N}}-\sum_{i=1}^{n}U_i(t)\underbrace{\int_0^R N_j(r)\mu\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial N_i(r)}{\partial r}2\pi r dr}_{\mathbf{K}}=0\end{aligned} \tag{4}$$

The subscript i is the summation index and index j corresponds j-th weight function.

Note: The change of weight function

$$w_j=\frac{dres}{dU_i} \tag{5}$$

And

$$w_j=1,r\ldots r^{j-1} \tag{6}$$

leads to the least square method and expert estimation of weight function method respectively.

It is more convenient to work with flow rates therefore we define total flow rate within the tube

$$q(t)=\sum_{i=1}^{n}U_i(t)2\pi\int_0^R rN_i(r)dr=\pi R^2\sum_{i=1}^{n}\frac{U_i(t)}{i(i+1)}=q_1(t)+q_2(t)+\ldots+q_n(t) \tag{7}$$

The subscript $i=1$ corresponds to Poiseuille flow while indices $i\geq 2$ to n denote the back flow computed from wall function.

$$q_{wi}(t)=2\pi U_i(t)\int_0^R rN_i(r)dr=\pi R^2\frac{U_i(t)}{i(i+1)}=q_{wi}(t) \tag{8}$$

It is simple to express the dependence the velocity $U_i(t)$ as function of flow rate now

$$\underbrace{\begin{bmatrix} U_1(t) \\ U_i(t) \\ \vdots \\ U_n(t) \end{bmatrix}}_{\mathbf{u}} = \pi R^2 \underbrace{\begin{bmatrix} 2 & -2 & -2 & -2 \\ 0 & i(i+1) & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & n(n+1) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} q(t) \\ q(t)_{wi} \\ \vdots \\ q(t)_{wn} \end{bmatrix}}_{\mathbf{q}} \tag{9}$$

The equations (4) can be rewritten in terms of flow rate in matrix form

$$\rho \frac{d\mathbf{q}}{dt} + \mathbf{M}^{-1}\mathbf{A}^{-1}\mathbf{N}P - \mu \mathbf{M}^{-1}\mathbf{A}^{-1}\mathbf{K}\mathbf{u} = 0 \tag{10}$$

Let us assume only the first term in polynomial series (3) (parabolic velocity profile) then q_{wi} is equal to zero. We obtain ordinary differential equation derived from (1) in the form

$$\rho \frac{dq}{dt} + \frac{2}{3} R^2 \pi P + \frac{16}{3} \frac{\mu}{R^2} q = 0 \tag{11}$$

The equation (11) leads in the case of stationary flow to the Hagen-Poiseuille flow. The introduction of the approximate solution of the pulsatile flow is used for more complicated situation with higher polynomial form of velocity profile. We consider the velocity profile as fourth order and sixth order of polynomial function, giving for two base functions (3)

$$\rho \begin{pmatrix} \frac{dq}{dt} \\ \frac{dq_{w2}}{dt} \end{pmatrix} + \pi R^2 \begin{pmatrix} \frac{8}{9} \\ \frac{5}{9} \end{pmatrix} P + \frac{\mu}{R^2} \begin{pmatrix} \frac{64}{9} & \frac{80}{9} \\ \frac{40}{9} & \frac{320}{9} \end{pmatrix} \begin{pmatrix} q \\ q_{w2} \end{pmatrix} = 0 \tag{12}$$

and for three terms

$$\rho \begin{pmatrix} \frac{dq}{dt} \\ \frac{dq_{w2}}{dt} \\ \frac{dq_{w3}}{dt} \end{pmatrix} + \pi R^2 \begin{pmatrix} \frac{15}{16} \\ -\frac{5}{12} \\ \frac{35}{48} \end{pmatrix} P + \frac{\mu}{R^2} \begin{pmatrix} \frac{15}{2} & 12 & \frac{51}{2} \\ -\frac{10}{3} & -\frac{80}{3} & -\frac{506}{3} \\ \frac{35}{6} & \frac{140}{3} & \frac{1015}{6} \end{pmatrix} \begin{pmatrix} q \\ q_{w2} \\ q_{w3} \end{pmatrix} = 0 \tag{13}$$

The reader will notice that the system of equations for different residual method and different integration varies in coefficients of ODE's and has to ask himself which method from class of weak formulation is near to the exact solution. The impact of the choice weighted residual method and the selection of integration will be discussed in the conclusion. The system (12), (13) satisfies Hagen-Poiseuille law when stationary flow is assumed.
The system (11) ,(12), (13)was solved with matlab ode113 solver (explicit Runge-Kutta 13th order) with absolute error set to 1e-8 for pulsatile flow with Womersley number $Wo = 1, 2, .., 11$ and 1e-9 with $Wo = 12, 13, .., 20$ that is defined by this way

$$Wo = R\sqrt{\frac{\omega\rho}{\mu}} \tag{14}$$

and it's the function of inner radius R and angular frequency ω.

2.2 Test example

The exact solution of (1) was computed by finite difference method using Matlab. The number of nodes on the radius was set to 1000. The driving force $P(t)$ was defined as harmonic function of Womersley number Wo ranges from 1 to 20.

$$P(t) = -1 + 2000\,sin(\omega t) \tag{15}$$

The amplitude of the oscillating pressure was selected so that the threshold of laminar flow was not exceeded ($\mathrm{Re} \approx 2300$).
The stationary flow is assumed at time $t = 0$

$$q(0) = -\frac{\pi R^4}{8\mu} P(0), q_{w2}(0) = 0 \tag{16}$$

The constants R , μ , ρ were set to correspond to the experiment.

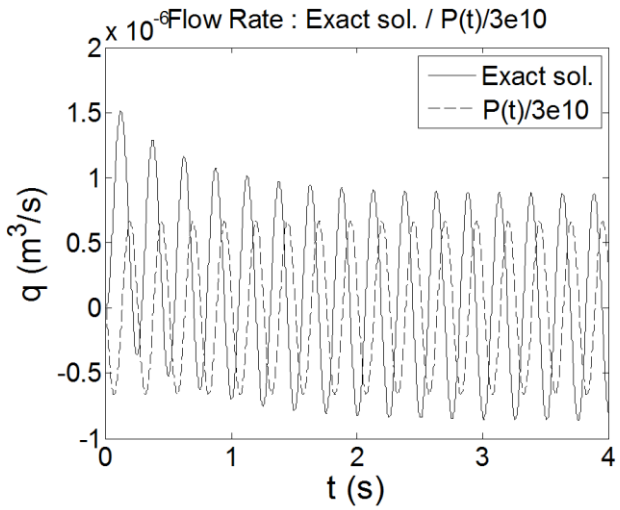

Fig. 1. *The example of exact solution of flow within the rigid tube is marked by black line whereas pressure gradient is scaled and labelled with dotted line. The tube has inner diameter 4 mm and it's considered an infinitely long. The water was assumed as oscillating fluid in the capillary.*

Situation presented in tested example can be solved analytically because we consider specific case of the driving force that has the form of trigonometric function. The situation starts to be more complicated in the case of arbitrary pressure variation in time (pumping action of the heart or the context of our experiment). The arbitrary pressure must be decomposed in a sum of sine and cosine functions known as Fourier series. The process of decomposition has to be repeated for any other time pressure variation. From this point of view, it's more suitable utilize the method of finite differences (the Fourier analysis isn't required). Moreover, the presented method of approximated solution simplifies partial differential equation (1) by the integral method to the system of ordinary two or three differential equations.

2.3 Determination of deviation

The deviation is computed as standard Euclidian norm of two functions of exact flow rate and approximated flow rates divided by a time $T = 4s$.

$$Norm\,q = \sqrt{\int_0^T \frac{1}{T}\left(q(t) - q(t)_{exact}\right)^2 dt} \tag{17}$$

3. Results

The errors of flow rates are presented on the figures bellow as norm (17) versus Womersley number. The total flow rate oscillates in order of 1e-6 m^3/s as can be seen in figure 1. The deviation of (11) and (12) is compared on the figure 2.

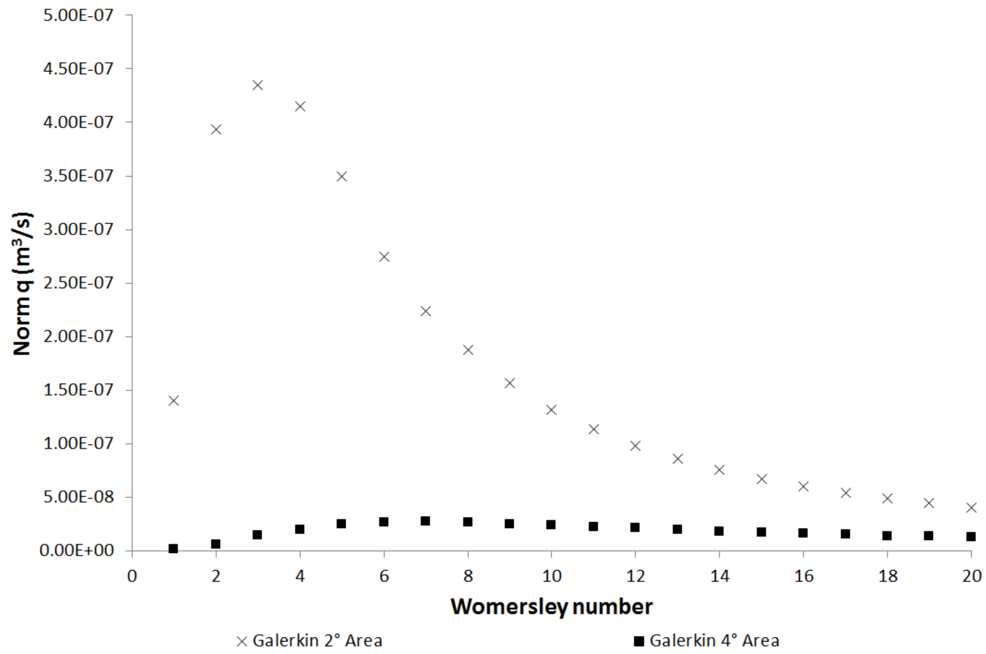

Fig. 2. *The error of flow rate computed from balance momentum equation approximated by parabolic velocity profile and fourth order polynomial velocity profile*

The figure 3 represents deviation of different residual method with different integration of residual and weight. The velocity profile with polynomial function of fourth order and sixth order was tested. The fulfilled diamonds represent the integration over area while the other the integration over the radius.

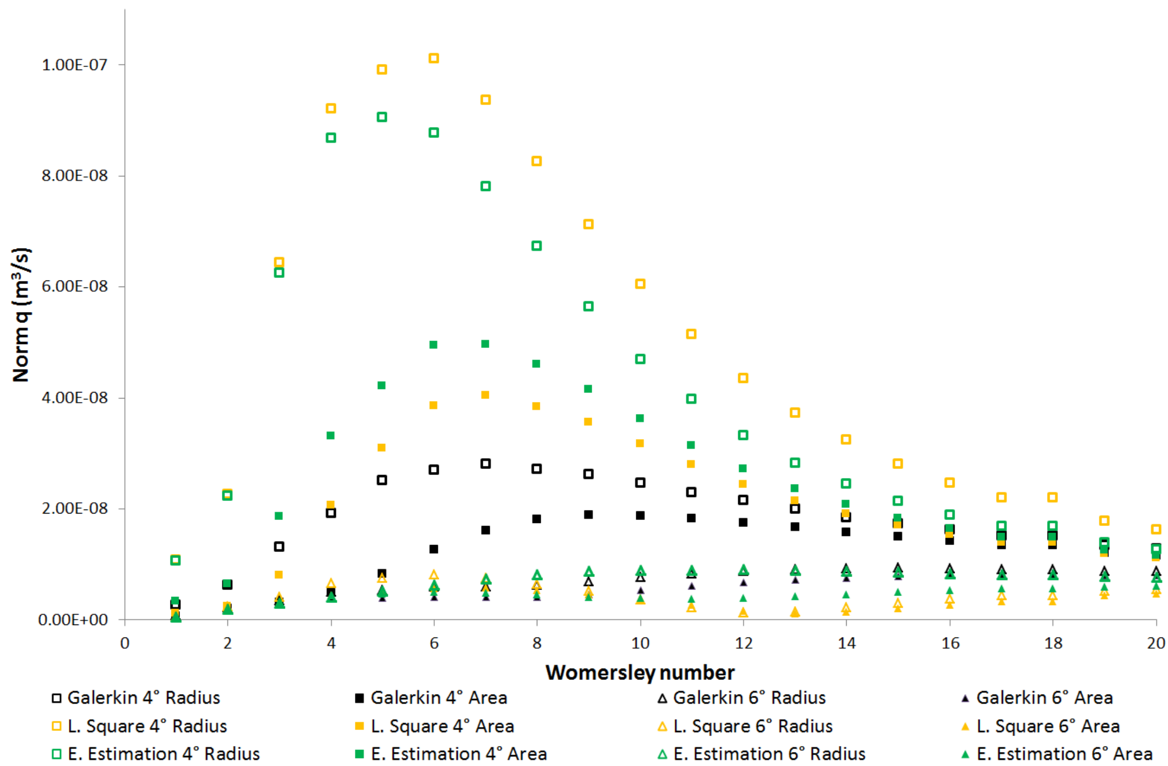

Fig. 3. *The error of flow rate from solving of ODE's derived from fourth order and sixth order polynomial velocity profile is presented. The Galerkin, least square and expert estimation method with integration over area and radius was tested.*

4. Conclusion

4.1 Comparison of parabolic velocity profile with exact solution

The pulsatile flow approximated with parabolic velocity profile gives error in tens of per cent in the region of Womersley number Wo =1-12. The solutions converge to the exact solution with rising Womersley numbers.

4.2 Comparison of velocity profile of polynomial function 4th and 6th order with exact solution

The error of approximate solution from balance of momentum with 4th order approximation of velocity profile is in units of per cent while the deviation evaluated from solution of 6th order of polynomial velocity profile is units of per mile. The flow rate computed from weak solution of balance momentum integrated over area computes the flow rate with smaller error then the integration of balance momentum over the radius. The integration over the area of the residual and weight puts more emphasis on the wall functions.
The analysis was performed for methods: Galerkin, least square method and the expert estimation of weight function ($w_1 = 1$ and $w_1 = r$). It was revealed that the smallest deviation from exact solution ensures the Galerkin method.

4.3 Recommendations for experimental device

The simulation was performed in same the region of pressure oscillations that correspond to the pressure oscillations of experiment and with the same inner diameter. The experiment oscillation coincides to $Wo \cong 10$. The reader can see that Hagen Poiseuille flow is inappropriate and that the most convenient method from class of weighted residual methods is the Galerkin with higher polynomial velocity profile (4th, 6th) with the integration over area.

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Symbols

ρ Water density $\left[kgm^{-3}\right]$

μ Dynamic viscosity $\left[Pas\right]$

ω Angular frequency $\left[s^{-1}\right]$

p Pressure $\left[Pa\right]$

x Axial coordinate $\left[m\right]$

u Fluid velocity $\left[ms^{-1}\right]$

r Radial coordinate $\left[m\right]$

R Radius of the tube $\left[m\right]$

P Pressure gradient $\left[Pam^{-1}\right]$

U_i Velocity amplitude $\left[ms^{-1}\right]$

N_i Basis function $\left[-\right]$

N_j j-th weighted function $\left[-\right]$

i i-th index

n n-th index

res Residual

w_j j-th weight function

q Total flow rate $\left[m^3s^{-1}\right]$

q_{wi} i-th flow rate near the wall $\left[m^3s^{-1}\right]$

Wo Womersley number $\left[-\right]$

$Norm\,q$ Deviation of flow rate $\left[m^3s^{-1}\right]$